CONSTRUCTION OF ANALYTIC FUNCTION

[Milne – Thomson method]

(i) To find $f(z)$ when u is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iu_y$ [by C–R condition]

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne–Thomson rule],

Where, C is a complex constant.

(ii) To find $f(z)$ when v is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= v_y + iv_x$ [by C–R condition]

$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne–Thomson rule],

Where, C is a complex constant.

Example: 3.22 Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$ $\quad$ [$\therefore \cos 0 = 1$]

$\Rightarrow u_x(z,0) = e^x$

$\Rightarrow u_y = e^x \cos y$ $\quad$ [$\therefore \sin 0 = 0$]

$\Rightarrow u_y(z,0) = 0$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne–Thomson rule],

Where, C is a complex constant.

$\therefore f(z) = \int e^z\, dz - i\int 0\, dz + C$

$= e^z + C$

Example: 3.23 Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solution:

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$u_x = e^{2x}[\cos 2y] + (xcos2y - y\sin 2y)[2e^{2x}]$$

$$u_x(z,0) = e^{2z}[1]+[z(1) - 0][2e^{2z}]$$

$$= e^{2z} + 2ze^{2z}$$

$$= (1+2z)e^{2x}$$

$$u_y = e^{2x}[-2x\sin 2y - (y2cos2y + \sin 2y)]$$

$$u_y(z,0) = e^{2z}[-0 - (0+0)] = 0$$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int(1+2z)e^{2z}dz - i\int 0 + dz + C$$

$$= \int(1+2z)e^{2z}dz + C$$

$$= (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + C \quad [\because \int uv\,dz = uv_1 - u'v_2 + u''v_3 - \cdots]$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + C$$

$$= ze^{2z} + C$$

Example: 3.24 Determine the analytic function where real part is
$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$. [Anna, May 2001]

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$\Rightarrow u_x(z,0) = 3z^2 - 0 + 6z$$

$$u_y = 0 - 6xy + 0 - 6y$$

$$\Rightarrow u_y(z,0) = 0$$

$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int(3z^2 + 6z)dz - i\int 0 + dz + C$$

$$= 3\frac{z^2}{3} + 6\frac{z^2}{2} + C$$

$$= z^3 + 3z^2 + C$$

Example: 3.25 Determine the analytic function whose real part in $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

[Anna, May 1996][A.U Tvli. A/M 2009][A.U N/D 2012]

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{[\cosh 2y - \cos 2x]^2}$$

$$u_x(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{[\cosh 0 - \cos 2z]^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2[\cos^2 2z + \sin^2 2z]}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)}$$

$$= \frac{-2}{2sin^2 2}$$

$$= -cosec^2 z$$

$$u_y = \frac{(\cosh 2y - \cos 2x)\,(0) - \sin 2x[2\sin 2y]}{[\cosh 2y - \cos 2x]^2}$$

$$\Rightarrow u_y(z,0) = 0$$

$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

where C is a complex constant.

$$f(z) = \int(-cosec^2 z)dz - i\int 0\,dz + C$$

$$= \cot z + C$$

Example: 3.26 Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate.

Also find $f(z)$ **[A.U A/M 2008, A.U A/M 2017 R8]**

Solution:

Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2},$$

$$\Rightarrow u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$u_{xx} = \frac{(x^2+y^2)[1]-x[2x]}{[x^2+y^2]^2} = \frac{x^2+y^2-2x^2}{[x^2+y^2]^2} = \frac{y^2-x^2}{[x^2+y^2]^2} \quad \ldots(1)$$

$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\Rightarrow u_y(z,0) = 0$$

$$u_{yy} = \frac{(x^2+y^2)[1]-y[2y]}{[x^2+y^2]^2} = \frac{x^2-y^2}{[x^2+y^2]^2} \quad \ldots(2)$$

To prove u is harmonic:

$$\therefore u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{[x^2+y^2]^2} = 0 \qquad by\ (1)\&(2)$$

$\Rightarrow u$ is harmonic.

To find $f(z)$:

$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne–Thomson rule],

Where, C is a complex constant.

$$f(z) = \int \frac{1}{z}dz - i\int 0\,dz + C$$

$$= log\, z + C$$

To find v :

$f(z) = log\,(re^{i\theta})$ $\qquad [\because z = re^{i\theta}]$

$$u + iv = \log r + \log e^{i\theta} = \log r + i\theta$$

$$\Rightarrow u = log r, v = \theta$$

Note: $z = x + iy$

$$r = |z| = \sqrt{x^2+y^2}$$

$$\log r = \frac{1}{2}\log(x^2+y^2)$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \qquad i.e., v = tan^{-1}\left(\frac{y}{x}\right)$$

Example: 3.27 Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v. **[A.U D15/J16, R-08]**

Solution:

Given $u = e^{x^2-y^2}\cos 2xy = e^{x^2}e^{-y^2}\cos 2xy$

$$u_x = e^{-y^2}\left[e^{x^2}(-2y\sin 2xy) + \cos 2xy\ e^{x^2}2x\right]$$

$$u_x(z,0) = 1\left[e^{z^2}(0) + 2ze^{z^2}\right] = 2ze^{z^2}$$

$$u_y = e^{x^2}\left[e^{-y^2}(-2x\sin 2xy) + \cos 2xye^{-y^2}(-2y)\right]$$

$$u_y(z,0) = e^{z^2}[0+0] = 0$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

$$= \int 2z\,e^{z^2}dz + C$$

$$= 2\int z\,e^{z^2}dz + C$$

put $t = z^2$, dt= $2z\,dz$

$$= \int e^t\,dt + C$$

$$= e^t + C$$

$$f(z) = e^{z^2} + C$$

To find v :

$$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}e^{i2\,xy}$$

$$= e^{x^2-y^2}\,[\cos(2xy) + i\sin(2xy)]$$

$$v = e^{x^2-y^2}\sin 2xy \quad [\because \text{equating the imaginary parts}]$$

Example: 3.28 Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$. **[Anna, May 1996] [A.U M/J 2014]**

Solution:

Given $v = e^{-x}(x\cos y + y\sin y)$

$$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$$

$$v_x(z,0) = e^{-z} + (z)(-e^{-z}) = (1-z)e^{-z}$$

$$v_y = e^{-x}[-x\sin y + (y\cos y + \sin y\,(1))]$$

$$v_x(z,0) = e^{-z}[0+0+0] = 0$$

$\therefore f(z) = \int v_y\,(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int(1-z)e^{-z}dz + C$$

$$= i\int(1-z)e^{-z}dz + C$$

$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$

$$= i[-(1-z)e^{-z} + e^{-z}] + C$$

$$= ize^{-z} + C$$

Example: 3.33 Determine the analytic function $f(z) = u + iv$ given that $3u + 2v = y^2 - x^2 + 16xy$ **[A.U. N/D 2007]**

Solution:

Given $3u + 2v = y^2 - x^2 + 16xy$ $\quad\ldots(A)$

Differentiate (A) p.w.r. to x, we get

$$3u_x + 2v_x = -2x + 16y$$

$3u_x - 2u_y = -2x + 16y$ [by C−R condition]

$$3u_x(z,0) - 2u_y(z,0) = -2z \quad\ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$3u_y + 2v_y = 2y + 16x$$

$3u_y + 2u_x = 2y + 16x$ [by C−R condition]

$$3u_y(z,0) + 2u_x(z,0) = 16z \quad\ldots(2)$$

$(1)\times(2) \quad \Rightarrow 6u_x(z,0) - 4u_y(z,0) = -4z \quad\ldots(3)$

$(2)\times(3) \quad \Rightarrow 9u_y(z,0) + 6u_x(z,0) = 48z$

$(3)-(4) \quad \Rightarrow -13u_y(z,0) = -52z$

$$\Rightarrow u_y(z,0) = 4z$$

$(1) \quad \Rightarrow 3u_x(z,0) = 8z - 2z = 6z$

$$\Rightarrow u_x(z,0) = 2z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

where C is a complex constant.

$$f(z) = \int 2zdz - i\int 4zdz + C$$

$$= 2\,\frac{z^2}{2} - i\frac{4z^2}{2} + C$$

$$= z^2 - i2z^2 + C$$

$$= (1-i2)z^2 + C$$

Example:3.34 Find an analytic function $f(z) = u + iv$ given that $2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$ [A.U. A/M 2017 R-8]

Solution:

Given $2u + 3v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$2u_x - 3u_y = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - \sin 2x\,(2\sin 2x)}{(\cosh 2y - \cos 2x)^2} \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = \frac{2\cos 2z(1 - \cos 2z) - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2} = \frac{-2}{1 - \cos 2z}$$

$$= \frac{-2}{2\sin^2 z} = -cosec^2 z$$

$$2u_x(z,0) - 3u_y(z,0) = -cosec^2 z \qquad \ldots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2)$$

$$2u_y + 3u_x = \frac{0 - \sin 2x(\sinh 2y)}{(\cosh 2y - \cos 2x)^2}(2) \quad \text{[by C − R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = 0 \qquad \ldots(2)$$

Solving (1) & (2) we get,

$$\Rightarrow u_x(z,0) = -\frac{2}{13}\mathrm{cosec}^2 z$$

$$\Rightarrow u_y(z,0) = -\frac{2}{13}\mathrm{cosec}^2 z$$

$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant

$$f(z) = \int\left(\frac{-2}{13}\right)\mathrm{cosec}^2 z\, dz - i\int\left(\frac{3}{13}\right)\mathrm{cosec}^2 z dz + C$$

$$= \left(\frac{2}{13}\right)\cot z + \left(\frac{3}{13}\right)\cot z + C$$

$$= \left(\frac{2}{13}\right)\cot z + \left(\frac{3}{13}\right)\cot z + C$$

$$= \frac{2+3i}{i3}\cot z + C$$

Example: 3.35 Find the analytic function $f(z) = u + iv$ given that $2u + 3v = e^x(\cos y - \sin y)$ [A.U A/M 22017 R-13]

Solution:

Given $2u + 3v = e^x(\cos y - \sin y)$

Differentiate p.w.r. to x, we get

$$2u_x + 3v_x = e^x(\cos y - \sin y)$$

$$2u_x - 3u_y = e^x(\cos y - \sin y) \quad \text{[by C–R condition]}$$

$$2u_x(z,0) - 3u_y(z,0) = e^z \qquad \ldots(1)$$

Differentiate p.w.r. to y, we get

$$2u_y + 3v_y = e^x[-\sin y - \cos y]$$

$$2u_y + 3u_x = -e^x\,[\sin y + \cos y] \quad \text{[by C–R condition]}$$

$$2u_y(z,0) + 3u_x(z,0) = -e^z \qquad \ldots(2)$$

$$(1) \times (3) \quad \Rightarrow 6u_x(z,0) - 9u_y(z,0) = 3e^z \qquad \ldots(3)$$

$$(2) \times 2 \quad \Rightarrow \; 6u_x(z,0) + 4u_y(z,0) = -2e^z \qquad \ldots(4)$$

$$(3) - (4) \quad \Rightarrow \; -13u_y(z,0) = 5e^z$$

$$\Rightarrow u_y(z,0) = -\frac{5}{13}e^z$$

$$(1) \quad \Rightarrow 2u_x(z,0) + \frac{15}{13}e^z = e^z$$

$$2u_x(z,0) = e^z - \frac{15}{13}e^z = -\frac{2}{13}e^z$$

$$\Rightarrow u_x(z,0) = -\frac{1}{13}e^z$$

$$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$$

$$\therefore f(z) = \int \frac{-1}{13}e^z\;dz - i\int\left(\frac{-5}{13}\right)dz + C$$

$$= \frac{-1}{13}e^z + \frac{5}{13}e^z i + C = \frac{-1+5i}{13}e^z + C$$