2.6 GAUSS DIVERGENCE THEOREM

This theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss Divergence theorem

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S, then

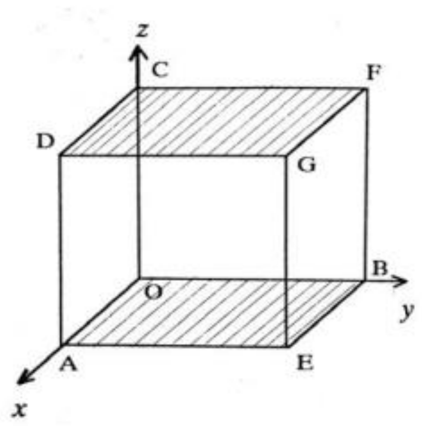
$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Where $\hat{n}$ is the unit outward normal to the surface S and $dV = dxdydz$

Problems based on gauss Divergence theorem

Example: 2.83 Verify the G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0,\ x = 1,$ $y = 0,\ y = 1,\ z = 0,\ z = 1.$

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\nabla \cdot \vec{F} = 4z - 2y + y$

$= 4z - y$

Now, R.H.S = $\iiint\limits_V \nabla \cdot \vec{F}\, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y) dxdydz$$

$$= \int_0^1 \int_0^1 [(4xz - yz)]_0^1 \; dydz$$

$$= \int_0^1 \int_0^1 (4z - y) dydz$$

$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \left(2 - \frac{1}{2}\right) - 0 = \frac{3}{2}$$

Now, L.H.S = $\iint\limits_S \vec{F} \cdot \hat{n}\, ds = \iint\limits_{S1} + \iint\limits_{S2} + \iint\limits_{S3} + \iint\limits_{S4} + \iint\limits_{S5} + \iint\limits_{S6}$

Faces	**Plane**	***dS***	$\hat{n}$	$\vec{F} \cdot \hat{n}$	**Equation**	$\vec{F} \cdot \hat{n}$ **on S**	$= \iint\limits_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z = 1$	y	$\int_0^1 \int_0^1 y\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	y^2	$y = 0$	0	$\int_0^1 \int_0^1 0\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$-y^2$	$y = 1$	-1	$\int_0^1 \int_0^1 -1 dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-4xz$	$x = 0$	0	$\int_0^1 \int_0^1 0\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$4xz$	$x = 1$	$4z$	$\int_0^1 \int_0^1 4z\, dydz$

$$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 y\, dxdy$$

$$= 0 + \int_0^1 \int_0^1 y\, dxdy$$

$$= \int_0^1 [yx]_0^1\, dy$$

$$= \int_0^1 y\, dy$$

$$= \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdz + \int_0^1 \int_0^1 -1\, dxdz$$

$$= 0 + \int_0^1 \int_0^1 -1\, dxdz$$

$$= -\int_0^1 [x]_0^1\, dz$$

$$= -\int_0^1 dz$$

$$= -[z]_0^1 = -[1]$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dydz + \int_0^1 \int_0^1 4z dydz$$

$$= 0 + \int_0^1 \int_0^1 4z\, dydz$$

$$= \int_0^1 [4zy]_0^1\, dz$$

$$= 4\left[\frac{z^2}{2}\right]_0^1 = 4\left(\frac{1}{2} - 0\right) = 2$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= \frac{1}{2} - 1 + 2 = \frac{3}{2}$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: 2.84 Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b, 0 \le z \le c$. **(OR)**

Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped bounded by $x = 0, x = a, y = 0, y = b, z = 0, z = c$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z \ = 2(x + y + z)$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= 2\int_0^c \int_0^b \int_0^a (x + y + z)dxdydz$$

$$= 2\int_0^c \int_0^b \left[\left(\frac{x^2}{2} + xy + xz\right)\right]_0^a dydz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dydz$$

$$= 2\int_0^c \left(\frac{a^2y}{2} + \frac{ay^2}{2} + azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + azb\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$$

$$= abc(a + b + c)$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	Eqn	$\vec{F} \cdot \hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-(z^2 - xy)$	$z = 0$	xy	$\int_0^b \int_0^a xy\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$(z^2 - xy)$	$z = c$	$c^2 - xy$	$\int_0^b \int_0^a c^2 - xy\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-(y^2 - xz)$	$y = 0$	xz	$\int_0^c \int_0^a xz\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$(y^2 - xz)$	$y = b$	$b^2 - xz$	$\int_0^c \int_0^a b^2 - xz\, dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(x^2 - yz)$	$x = 0$	yz	$\int_0^c \int_0^b yz\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(x^2 - yz)$	$x = a$	$a^2 - yz$	$\int_0^c \int_0^b a^2 - yz\, dydz$

$$(i)\iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dxdy + \int_0^b\int_0^a c^2 - xy\,dxdy$$

$$= \int_0^b\int_0^a c^2\,dxdy$$

$$= c^2\int_0^a dx\int_0^b dy$$

$$= c^2[x]_0^a[y]_0^b = c^2ab$$

$$(ii)\iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a xz\,dxdz + \int_0^c\int_0^a b^2 - xz\,dxdz$$

$$= \int_0^c\int_0^a b^2\,dxdz$$

$$= b^2\int_0^a dx\int_0^c dz$$

$$= b^2[x]_0^a[z]_0^c = b^2ac$$

$$(iii)\iint_{S5} \vec{F}\cdot\hat{n}\,ds + \iint_{S6} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b yz\,dydz + \int_0^c\int_0^b a^2 - yz\,dydz$$

$$= \int_0^c\int_0^b a^2\,dydz$$

$$= a^2\int_0^b dy\int_0^c dz$$

$$= a^2[y]_0^b[z]_0^c = a^2bc$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= abc(a+b+c)$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Hence Gauss divergence theorem is verified.

Example: 2.85 Verify divergence theorem for $\vec{F} = (2x - z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (2x - z)\vec{i} + x^2y\vec{j} - xz^2\vec{k}$

$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz)\, dxdydz$

$= \int_0^1 \int_0^1 \left[\left(2x + \frac{x^3}{3} - \frac{2zx^2}{2}\right)\right]_0^1 dydz$

$= \int_0^1 \int_0^1 \left(2 + \frac{1}{3} - z\right) dydz$

$= \int_0^1 \left(2y + \frac{1}{3}y - zy\right)_0^1 dz$

$= \int_0^1 \left(2 + \frac{1}{3} - z\right) dz$

$= \left[2z + \frac{1}{3}z - \frac{z^2}{2}\right]_0^1$

$= \left(2 + \frac{1}{3} - \frac{1}{2}\right) - 0 = \frac{11}{6}$

Now, L.H.S = $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	***dS***	$\hat{n}$	$\vec{F}\cdot\hat{n}$	**Equation**	$\vec{F}\cdot\hat{n}$ **on S**	$= \iint_S \vec{F}\cdot\hat{n}\,ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	xz^2	$z=0$	0	$\int_0^1\int_0^1 0\,dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$-xz^2$	$z=1$	$-x$	$\int_0^1\int_0^1 (-x)\,dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-x^2y$	$y=0$	0	$\int_0^1\int_0^1 0\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	x^2y	$y=1$	x^2	$\int_0^1\int_0^1 x^2dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(2x-z)$	$x=0$	z	$\int_0^1\int_0^1 z\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(2x-z)$	$x=1$	$2-z$	$\int_0^1\int_0^1 2-z\,dydz$

$(i) \iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 (-x)\,dxdy$

$= \int_0^1\int_0^1 (-x)\,dxdy$

$$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= -\int_0^1 \frac{1}{2}\, dy$$

$$= -\left[\frac{1}{2}y\right]_0^1 \quad = -\left(\frac{1}{2} - 0\right) \quad = \frac{-1}{2}$$

$$(ii) \iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdz + \int_0^1\int_0^1 x^2\,dxdz$$

$$= \int_0^1\int_0^1 x^2\,dxdz$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz$$

$$= \int_0^1 \frac{1}{3}\, dz$$

$$= \left[\frac{1}{3}z\right]_0^1 \quad = \left(\frac{1}{3} - 0\right) \quad = \frac{1}{3}$$

$$(iii) \iint_{S5} \vec{F}\cdot\hat{n}\,ds + \iint_{S6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 z\,dydz + \int_0^1\int_0^1 (2-z)\,dydz$$

$$= \int_0^1\int_0^1 2\,dydz$$

$$= 2\int_0^1 [y]_0^1\, dz$$

$$= 2\int_0^1 dz$$

$$= 2\,[z]_0^1 = 2$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= -\frac{1}{2} + \frac{1}{3} + 2 \quad = \frac{11}{6}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Hence Gauss divergence theorem is verified.

Exercise: 2.5

1. Verify divergence theorem for the function $\vec{F} = (x^2 - yz)\vec{\imath} - (y^2 - zx)\vec{\jmath} + (z^2 - xy)\vec{k}$ over the surface bounded by $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** 36
2. Verify divergence theorem for the function $\vec{F} = 4xz\vec{\imath} - y^2\vec{\jmath} + yz\vec{k}$ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ **Ans:** Common value $= \frac{3}{2}$
3. Verify divergence theorem for the function $\vec{F} = (2x - z)\vec{\imath} - x^2y\vec{\jmath} - xz^2\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$ **Ans:** Common value $= \frac{11}{6}$
4. Verify divergence theorem for $\vec{F} = xy^2\vec{\imath} + yz^2\vec{\jmath} + zx^2\vec{k}$ over the region $x^2 + y^2 = 4 \text{ and } z = 0, z = 3$ **Ans:** Common value $= 84\pi$
5. Using divergence theorem, prove that (i) $\iint_S \vec{R} \cdot d\vec{S} = 3V$ (ii) $\iint_S \nabla r^2 \cdot d\vec{S} = 6V$
6. $\vec{F} = x^2\vec{\imath} + z\vec{\jmath} + yz\vec{k}$ over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$ **Ans:** Common value $= \frac{3a}{2}$
7. $\vec{F} = (x^3 - yz)\vec{\imath} - 2x^2y\vec{\jmath} + 2\vec{k}$ over the parallelopiped bounded by the planes $x = 0, x = 1, y = 0, y = 2, z = 0, z = 3$ **Ans:** Common value $= 2$