CHAPTER 3

Transient Response Analysis

3.1 LAPLACE TRANSFORM

Learning Objective (LO 1)

- ***Students will be able to enumerate the basic concept of Laplace transform.***

- The Laplace transform is used to solve differential equations and corresponding initial and final value problems.
- Laplace transforms are widely used in engineering particularly when the driving function has discontinuities and appears for a short period only.
- In circuit analysis, the input and output functions do not exist forever in time. For casual functions, the function can be defined as $f(t)\,u(t)$.
- The integral for the Laplace transform is taken with the lower limit at $t = 0$ in order to include the effect of any discontinuity at $t = 0$.
- Consider a function $f(t)$ which is to be continuous and defined for values of $t \geq 0$.
- The Laplace transform is then

$$L[f(t)] = F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, u(t)\, dt = \int_{0}^{\infty} e^{-st}\, dt$$

- $f(t)$ is a continuous function for $t \geq 0$ multiplied by e^{-st} which is integrated with respect to between limits 0 and ∞.
- The resultant function for the variable is called Laplace transform of $f(t)$.
- Laplace transform is a function of independent variables corresponding to the complex variable in the exponent of e^{-st}.

- The complex variable s is, in general of the form $s = \sigma + j\omega$ ... the real and imaginary parts, respectively.
- For a function to have a Laplace transform, it must satisfy

$$\int_0^{\infty} f(t)\, e^{-st}\, dt < \infty$$

- Laplace transform changes the time domain function $f(t)$ ... domain function $F(s)$.

Note

- If we can use these two properties jointly, we have

$$L[K_1 f_1(t) + K_2 f_2(t)] = K_1 L[f_1(t)] + K_2 L[f_2(t)]$$

$$= K_1 F_1(s) + K_2 F_2(s)$$

Properties of Laplace Transforms (Formulas)

1. $L[Af(t)] = AF(s)$
2. $L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3. $L\left[\frac{d}{dt} f(t)\right] = sF(s) - F(0)$
4. $L\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \left.\frac{d^{k-1} f(t)}{dt^{k-1}}\right|_{t=0}$
5. $L\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\, dt\right]_{t=0}}{s}$
6. $L[e^{-at} f(t)] = F(s+a)$
7. $L[f(t-\alpha)\, u(t-\alpha)] = e^{-\alpha s} F(s)\,;\ \alpha = 0$
8. $L[tf(t)] = \frac{-dF(s)}{ds}$
9. $L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$ $\quad n = 1, 2, 3, \ldots$
10. $L\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

1.3 Laplace Transform Formulas

Sl.No	$f(t)$	$F(s)$
1.	unit step, $u(t)$	$\frac{1}{s}$
2.	unit impulse $\delta(t)$	1
3.	t	$\frac{1}{s^2}$
4.	$\frac{t^{n-1}}{(n-1)!}$ $(n = 1, 2, 3 \ldots)$	$\frac{1}{s^n}$
4.1	$e^{\omega t}$	$\frac{1}{s-\omega}$
5.	$e^{-\omega t}$	$\frac{1}{s+\omega}$
6.	t^n	$\frac{n!}{s^n + 1}$
7.	$te^{-\omega t}$	$\frac{1}{(s+\omega)^2}$
8.	$\frac{1}{(n-1)!} t^{n-1} e^{-\omega t}$	$\frac{1}{(s+a)^n}$
9.	$t^n e^{-\omega t}$ $(n = 1, 2, 3 \ldots)$	$\frac{n!}{(s+\omega)^{n+1}}$
10.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12.	$\sinh \omega t$	$\frac{\omega}{s^2+\omega^2}$
13.	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14.	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a^2)^2+\omega^2}$

Sl.No	$f(t)$	$F(s)$
15.	$e^{\omega t} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$
16.	$e^{at}\, t^n$	$\frac{n!}{(S+a)^{n+1}}$
17.	$e^{at}\, t^n$	$\frac{n!}{s^{n+1}}$
18.	$e^{at} \sinh \omega t$	$\frac{\omega}{(s-2)^2-\omega^2}$
19.	$e^{at} \cosh \omega t$	$\frac{s-a}{(s-a)^2-\omega^2}$

EXAMPLE 1: *Find the Laplace transform of the function* $f(t) = 8t^3 + 2t^2 - 8t + 16$

LOTS

Solution :

$$L\,(8t^3 + 2t^2 - 6t + 6) = 8L\,(t^3) - 2L\,(t^2) + 8L\,(t) + 16L\,(1)$$

$$= 8 \times \frac{3!}{s^4} + \frac{2 \times 2!}{s^3} - 8\,\frac{1!}{s^2} + 16\,\frac{1}{s}$$

$$= \frac{48}{s^4} + \frac{4}{s^3} - \frac{88}{s^2} + \frac{16}{s}$$

EXAMPLE 2: *Find the Laplace transform of the function* $f(t) = \sin^2 t$

LOTS

Solution :

$$L\left(\frac{1 - \cos 2t}{2}\right)$$

$$L\left(\frac{1}{2}\right) - L\,\frac{\cos 2t}{2}$$

$$= \frac{1}{2s} - \frac{s}{2\,(s^2+4)} = \frac{2s^2 - 4}{2s\,(s^2+4)}$$

EXAMPLE 3: *Find the Laplace transform of the function.*

$$f(t) = 8t^4 + 10t^3 + 12e^{-3t} - 14 \sin 5t + 16 \cos 2t$$

LOTS

Solution :

$$Lf(t) = 8L(t^4) - 10L(t^3) + 12L(e^{-3t}) - 14L(\sin 5t) + 16L \cos 2t$$

$$= \frac{8 \times 4!}{s^5} - \frac{10 \times 3!}{s^4} + \frac{12 \times 1}{s+3} - \frac{14 \times 5}{s^2+25} + \frac{16 - s}{s^2+4}$$

$$F(s) = \frac{192}{s^5} - \frac{60}{s^4} + \frac{12}{s+3} - \frac{70}{s^2+25} + \frac{16s}{s^2+4}$$

3.2 INVERSE LAPLACE TRANSFORMATION

Learning Objective (LO 2)

- ***Students will be able to explain the basic concept of inverse Laplace transform.***

- Inverse Laplace transform converts frequency domain function $F(s)$, to the time domain function $f(t)$ as shown below.

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{-j}^{+j} F(s)\, e^{st}\, ds$$

- Here, the inverse transform involves a complex integration. $f(t)$ can be represented as a weighted integral of complex exponentials.
- We will denote the transform relationship between $f(t)$ and $F(s)$ as

$$f(t) \xrightarrow{L} F(s)$$

3.2.1 Inverse Laplace Transform Formulas

Sl.No.	$F(s)$	$f(t)$
1.	$\frac{1}{s}$	$u(t)$
2.	$\frac{1}{s^2}$	$tu(t)$
3.	1	$\delta(t)$ impulse function

4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s+a}$	e^{-at}
6.	$\frac{1}{(s-a)^2}$	te^{at}
7.	$\frac{a}{s^2+a^2}$	$\sin at$
8.	$\frac{s}{s^2+a^2}$	$\cos at$
9.	$\frac{a}{s^2-a^2}$	$\sinh at$
10.	$\frac{s}{s^2-a^2}$	$\cosh at$
11.	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$ where $n = 1, 2, 3, \ldots$
12.	$\frac{n!}{s^{n+1}}$	t^n where $n = 1, 2, 3 \ldots$
13.	$\frac{b}{(s-a)^2+b^2}$	$e^{at} \sin bt$
14.	$\frac{(s-a)}{(s-a)^2+b^2}$	$e^{at} \cos bt$
15.	$\frac{b}{(s-a)^2-b^2}$	$e^{at} \sinh bt$
16.	$\frac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$
17.	$\frac{1}{(s-a)^n}$	$\frac{1}{(n-1)\,1} t^{n-1} \cdot e^{at}$
18	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}\, t^n$