$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_1\,\omega_2 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)\left(\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right)$$

$$= -\left(\frac{R}{2L}\right)^2 + \left(-\frac{R}{2L}\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right) + \left(\frac{R}{2L}\sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}\right) + \left(\frac{R}{2L}\right)^2 + \frac{1}{LC}$$

$$= -\left(\frac{R}{2L}\right)^2 + \left(\frac{R}{2L}\right)^2 + \frac{1}{LC} = \frac{1}{LC} = \left(\frac{1}{\sqrt{LC}}\right)^2 = \omega_r^2$$

$$\boxed{\omega_1\omega_2 = \omega_r^2}$$

4.3 COUPLED CIRCUITS

Learning Objective (LO 3)

- ***Students will be able to derive the expression for self inductance, mutual inductance, coefficient of coupling in coupled circuits.***

Two circuits are said to be coupled when energy transfer takes place from one circuit to the other when one of the circuits is energised. The coupling may be conductive, inductive or magnetic.

When two coils are placed nearby and current passes through any one or both of the coils, they become magnetically coupled. Then the coils are known as coupled coils. If the coils are part of a circuit, the circuit is known as a coupled circuit.

4.3.1 Self Inductance

Figure 4.11 shows a coil of N turns carrying current i. ϕ is the flux produced in webers. If the current i through the coil changes, the flux ϕ also changes. By Faraday's law, change in flux induces an emf in the coil.

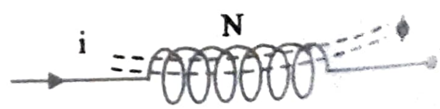


Fig. 4.11

The induced emf is proportional to rate of change of flux.

$$e \propto \frac{d\phi}{dt} \qquad \text{...(1)}$$

$$e = -\frac{N\,d\phi}{dt}$$

The negative sign indicates that the emf induced is opposite to flux causing it. The induced emf is also proportional to the rate of change of current causing the flux ϕ.

$$e \propto \frac{di}{dt}$$

$$e = -L\frac{di}{dt} \quad ...(2)$$

where L is the self inductance of the coil. The negative sign indicates that the emf induced is opposite to the current causing it.

Equating (1) and (2)

$$-N\frac{d\phi}{dt} = -L\frac{di}{dt}$$

$$N\,d\phi = L\,di$$

$$L = N\frac{d\phi}{di}$$

For constant permeability

$$L = \frac{N\phi}{i}$$

Self inductance of a coil is defined as the flux linkage ($N\phi$) produced in the coil per unit current in the same coil. The unit is henry.

4.3.2 Mutual Inductance

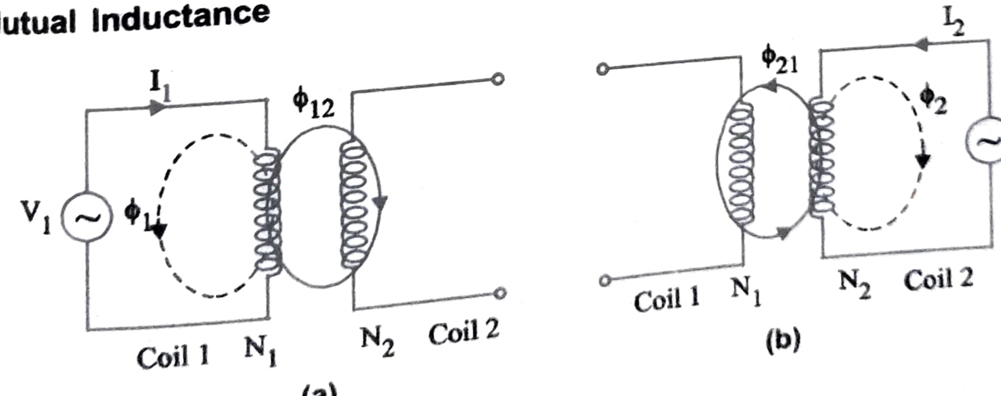


Fig. 4.12

Let N_1 be the number of turns in coil 1. N_2 be the number of turns in coil 2. If a varying voltage source V_1 is connected to coil 1, it produces varying current I_1. The varying current I_1 produces a changing flux ϕ_1 in it. It is shown in figure 4.12 (a)

The flux ϕ_1 is divided into two parts

$$\phi_1 = \phi_{11} + \phi_{12}$$

where ϕ_{11} is the part of flux ϕ_1, which links only with coil 1. ϕ_{12} is the part of flux ϕ_1 which links with both the coils 1 & 2. Let the induced emf in the coil 2 be e_2. The induced emf e_2 is proportional to the rate of change of flux in the coil 2.

$$e_2 \propto \frac{d\phi_{12}}{dt}$$

$$e_2 = -N_2 \frac{d\phi_{12}}{dt} \quad \text{...(3)}$$

The induced emf is also proportional to the rate of change of current causing the flux ϕ_2. The current causing the flux ϕ_{12} is i_1

$$e_2 \propto \frac{di_1}{dt}$$

$$e_2 = -M \frac{di_1}{dt} \quad \text{...(4)}$$

Equating (3) and (4)

$$-N_2 \frac{d\,\phi_{12}}{dt} = -M \frac{di_1}{dt}$$

$$N_2\, d\,\phi_{12} = M\, di_1$$

$$M = N_2 \frac{d\,\phi_{12}}{di_1}$$

If permeability is constant,

$$\boxed{M = \frac{N_2\,\phi_{12}}{i_1}} \quad \text{.... (5)}$$

If a varying voltage source V_2 connected to coil 2 produces a varying current I_2. The varying current I_2 in coil 2 produces a change in flux ϕ_2 in it. It is shown in figure 4.12 (b). The flux ϕ_2 is divided into two parts.

$$\phi_2 = \phi_{22} + \phi_{21}$$

where ϕ_{22} is the part of flux ϕ_2 linking only with coil 2 and ϕ_{21} is the part of flux ϕ_2 linking with coil 2 and coil 1. Let the induced emf in coil 1 be e_1. The induced emf e_1 is proportional to rate of change of flux in coil 1.

$$e_1 \propto \frac{d\,\phi_{21}}{dt}$$

$$e_1 = -N_1 \frac{d\,\phi_{21}}{dt} \qquad ...(6)$$

The induced emf is also proportional to the rate of change of current causing the flux ϕ_{21}. The current causing the flux ϕ_{21} is i_2.

$$e_1 \propto \frac{di_2}{dt}$$

$$e_1 = -M \frac{di_2}{dt} \qquad ...(7)$$

Equating (6) and (7)

$$-N_1 \frac{d\,\phi_{21}}{dt} = -M \frac{di_2}{dt}$$

$$M = N_1 \frac{d\phi_{21}}{di_2}$$

If permeability is constant,

$$\boxed{M = \frac{N_1\,\phi_{21}}{i_2}} \qquad .... (8)$$

Mutual inductance is defined as the ability of varying current in one coil to produce an induced emf in the other coil. Its unit is henry.

Mutual inductance between two circuits is defined as the flux linkages of one circuit per unit current in the other circuit.

4.3.3 Co-efficient of Coupling (K) (or) Magnetic Coupling

Co-efficient of coupling is defined as the fraction of the total flux produced by one coil linking the other coil.

$$\therefore \qquad K = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{21}}{\phi_2}$$

We know $\quad M = \dfrac{N_2\,\phi_{12}}{i_1}$ and $M = \dfrac{N_1\,\phi_{21}}{i_2}$

$$\therefore \qquad M^2 = \left(\frac{N_2\,\phi_{12}}{i_1}\right)\left(\frac{N_1\,\phi_{21}}{i_2}\right)$$

$$M^2 = \frac{N_1\, N_2\, \phi_{12}\, \phi_{21}}{i_1\, i_2}$$

where $\phi_{12} = K\,\phi_1$ and $\phi_{21} = K\,\phi_2$

$\therefore$
$$M^2 = \frac{N_2\, K\, \phi_1\, N_1\, K\, \phi_2}{i_2\, i_2}$$

or
$$M^2 = \frac{K^2\, N_2\, N_1\, \phi_1\, \phi_2}{i_2\, i_2}$$

$$M^2 = K^2 \left(\frac{N_1\, \phi_1}{i_1}\right)\left(\frac{N_2\, \phi_2}{i_2}\right)$$

$$M^2 = K^2\, L_1\, L_2$$

$$M = K\sqrt{L_1\, L_2}$$

$$\boldsymbol{K = \frac{M}{\sqrt{L_1\, L_2}}} \quad \text{.... (9)}$$

The value of K depends on the spacing between the coils. As spacing increases, K decreases. K is always positive and its maximum value is 1.

The maximum mutual inductance occurs when $K = 1$ and so

$$\boldsymbol{M_{max} = \sqrt{L_1\, L_2}} \quad \text{.... (10)}$$

EXAMPLE 40: *Obtain the maximum possible mutual inductance between two coils of inductance 16 H and 4 H.*

LOTS

Solution :

$$M = K\sqrt{L_1\, L_2}$$

For maximum possible mutual inductance, $K = 1$

$$L_1 = 16\text{ H}, L_2 = 4\text{ H}$$

$$M_{max} = \sqrt{L_1\, L_2} = \sqrt{16 \times 4} = \sqrt{64}$$

$$\boldsymbol{M = 8\text{ H}}$$

EXAMPLE 41: *Two inductively coupled coils have self inductance $L_1 = 50\ mH$ and $L_2 = 200\ mH$. If the coefficient of coupling is 0.5 (i) Find the value of mutual inductance between the coils and (ii) What is the maximum possible mutual inductance?*

LOTS

Solution :

$L_1 = 50\text{ mH}, L_2 = 200\text{ mH}$

(i) $K = 0.5$; $M = K\sqrt{L_1 L_2} = 0.5\sqrt{50 \times 10^{-3} \times 200 \times 10^{-3}}$

$$\boxed{M = 50 \times 10^{-3}\text{ H}}$$

(ii) Maximum value of the inductance occurs when $K = 1$

$$M = \sqrt{L_1 L_2} = \sqrt{50 \times 10^{-3} \times 200 \times 10^{-3}}$$

$$\boxed{M = 100 \times 10^{-3}\text{ H}}$$

EXAMPLE 42: *Two identical coils A and B of 1000 turns each lie in parallel planes such that 80% flux produced by one coil links with the other coil. A current of 5 A flowing in coil A produces a flux of 0.05 mWb in it. If the current in coil A changes from + 12 A to − 12 A in 0.02 s, calculate. (i) the mutual inductance (ii) the emf induced in the coil B*

LOTS

Solution :

$N_1 = N_2 = 1000$ turns, $\phi_1 = 0.05 \times 10^{-3}$ Wb

$\phi_{12} = 0.8 \times 0.05 \times 10^{-3} \Rightarrow 0.04 \times 10^{-3}$ Wb

(i) Mutual inductance

$$M = \frac{N_2\,\phi_{12}}{i_1} = \frac{1000 \times 0.04 \times 10^{-3}}{5}$$

$$\boxed{M = 8 \times 10^{-3}\text{ H}}$$

(ii) EMF induced in the coil B

$$e_2 = M\frac{di_1}{dt}$$

$$di_1 = 12 - (-12) = 24\text{ A}$$

$$e_2 = \frac{8 \times 10^{-3} \times 24}{0.02} = 9.6\text{ volts}$$

$$\boxed{e_2 = 9.6\text{ V}}$$

EXAMPLE 43: *The number of turns in two coupled coils are 500 turns and 1500 turns respectively. When 5A current flows in coil 1, the total flux in this coil is 0.6×10^{-3} Wb and the flux linking the second coil is 0.3×10^{-3} Wb. Determine L_1, L_2, M and K.*

HOTS

Solution :

$N_1 = 500$ turns, $N_2 = 1500$ turns, $i_1 = 5$ A, $\phi_{11} = 0.6 \times 10^{-3}$ Wb

$$\phi_{12} = 0.3 \times 10^{-3} \text{ wb}$$

$$M = \frac{N_2 \phi_{12}}{i_1} = \frac{1500 \times 0.3 \times 10^{-3}}{5}$$

$$\boxed{M = 0.09 \text{ H}}$$

$$\phi_1 = \phi_{12} + \phi_{11} = 0.6 \times 10^{-3} + 0.3 \times 10^{-3}$$

$$\phi_1 = 0.9 \times 10^{-3} \text{ wb}$$

$$L_1 = \frac{N_1 \phi_1}{i_1} = \frac{500 \times 0.9 \times 10^{-3}}{5}$$

$$\boxed{L_1 = 0.09 \text{ H}}$$

$$K = \frac{\phi_{12}}{\phi_1} = \frac{0.3 \times 10^{-3}}{0.9 \times 10^{-3}} = 0.333$$

$$M = K\sqrt{L_1 L_2}$$

$$M^2 = K^2 L_1 L_2$$

$$L_2 = \frac{M^2}{K^2 L_1} = \frac{(0.09)^2}{(0.333)^2 \times 0.09}$$

$$\boxed{L_2 = 0.81 \text{ H}}$$

4.3.4 Dot Rule

For analysing coupled circuits, coils are marked with dots as shown in figure 4.13.

1. If both currents enter or leave the coils through the dotted ends of the coil, then the sign of M is positive.
2. If one current enters a coil through the dotted end and the other leaves the coil through the dotted end, then the sign of M is negative.

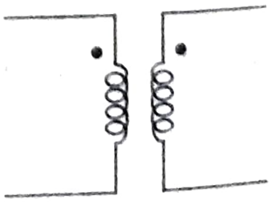


Fig. 4.13

Figure 4.14 shows coupled circuits having positive mutual inductance

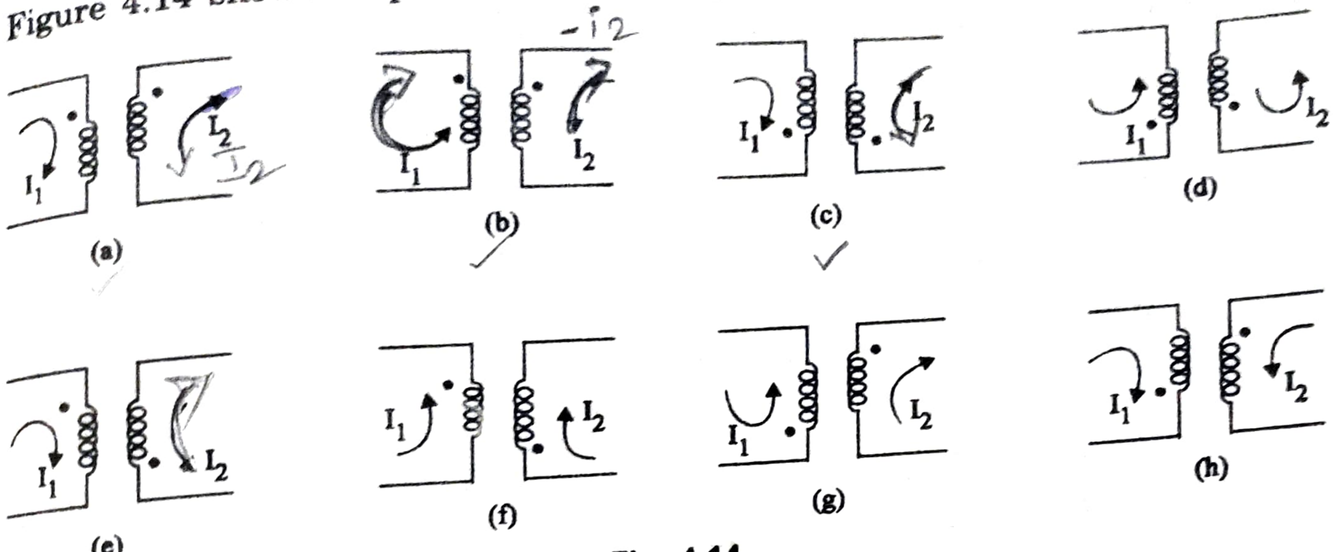


Fig. 4.14

Figure 4.15 shows coupled circuits having negative mutual inductance.

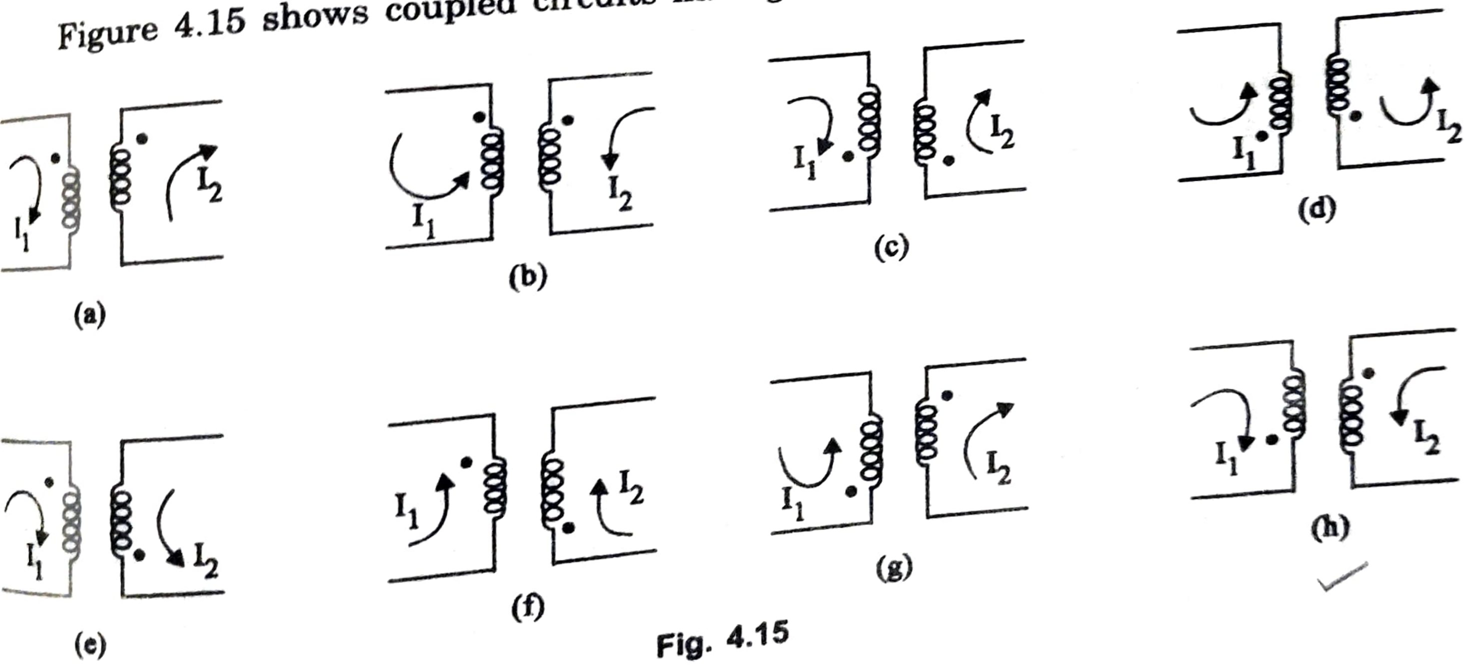


Fig. 4.15

4.3.5 Conductively Coupled Coils

Two coils of self inductances L_1 and L_2 and having a mutual inductance M between them can be connected in four different ways.

(i) Series aiding (ii) Series opposing (iii) Parallel aiding (iv) Parallel opposing

4.3.5.1 Series aiding

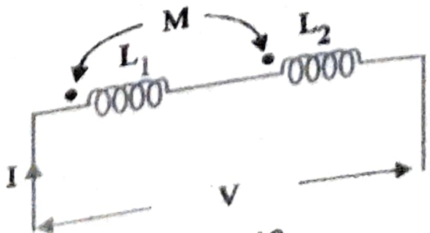


Fig. 4.16

Figure 4.16 shows two inductors connected in series aiding.

The equivalent inductance is given by

$$L_{eq} = L_1 + L_2 + 2M$$

$$L = \frac{\phi}{I}$$

$$\phi = LI$$

$$\phi = \phi_1 + \phi_2$$

$$\phi_1 = L_1 I_1 + M I_2$$

$$\phi_2 = L_2 I_2 + M I_1$$

$$\phi = LI = L_1 I_1 + M I_2 + L_2 I_2 + M I_1$$

Here, $I_1 = I_2 = I$

$$\boxed{\therefore L = L_1 + L_2 + 2M} \quad \dots (11)$$

4.3.5.2 Series opposing

Figure 4.17 shows two inductors connected in series opposing.

The equivalent inductance is given by

$$L_{eq} = L_1 + L_2 - 2M$$

$$\phi = LI$$

$$\phi_1 = L_1 I_1 - M I_2$$

$$\phi_2 = L_2 I_2 - M I_1$$

$$\phi = LI = L_1 I_1 - M I_2 + L_2 I_2 - M I_1$$

$$I_1 = I_2 = I$$

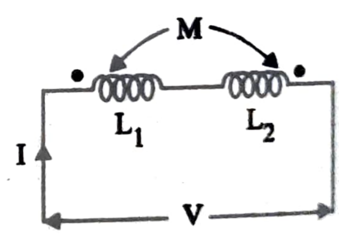


Fig. 4.17

$$\boxed{L = L_1 + L_2 - 2M} \quad \dots (12)$$

4.3.5.2 Parallel aiding

Figure 4.18 shows two inductors connected in parallel aiding.

The equivalent inductance is given by

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

$$j\omega L_1 I_1 + j\omega M I_2 = V$$

$$j\omega M I_1 + j\omega L_2 I_2 = V$$

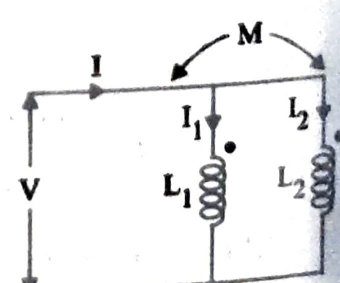


Fig. 4.18

Solving we get the loop currents I_1 and I_2 as

$$I_1 = \frac{j\,\omega\,(L_2 - M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

$$I_2 = \frac{j\,\omega\,(L_1 - M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

The total current drawn from the source is given by

$$I = I_1 + I_2 = \frac{j\,\omega\,(L_1 + L_2 - 2M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

The equivalent impedance Z is given by

$$Z = \frac{V}{I} = \frac{\omega^2\,(M^2 - L_1 L_2)}{j\,\omega\,(L_1 + L_2 - 2M)} = \frac{j\,\omega\,(L_1 L_2 - M^2)}{L_1 + L_2 - 2M}$$

The equivalent inductance is given by

$$\boxed{L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}} \quad \text{... (13)}$$

4.3.5.3 Parallel opposing

Figure 4.19 shows the two inductances are connected in parallel opposing.

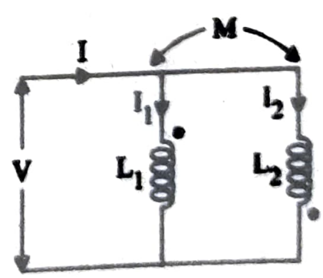


Fig. 4.19

The equivalent inductance is given by

$$j\,\omega L_1 I_1 - j\,\omega M I_2 = V$$

$$-j\,\omega M I_1 + j\,\omega L_2 I_2 = V$$

Solving we get the loop current I_1 and I_2 as

$$I_1 = \frac{j\,\omega\,(L_2 + M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

$$I_2 = \frac{j\,\omega\,(L_1 + M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

The total current drawn from the source is given by

$$I = I_1 + I_2 = \frac{j\,\omega\,(L_1 + L_2 + 2M)\,V}{\omega^2\,(M^2 - L_1 L_2)}$$

The equivalent impedance Z is given by

$$Z = \frac{V}{I} = \frac{\omega^2 (M^2 - L_1 L_2)}{j\,\omega\,(L_1 + L_2 + 2M)}$$

$$Z = \frac{j\,\omega\,(L_1 L_2 - M^2)}{L_1 + L_2 + 2M}$$

The equivalent inductance is given by

$$L_{eq} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M} \qquad \dots (14)$$

EXAMPLE 44: *Two coils connected in series have an equivalent inductance of 0.8 H when connected in aiding and an equivalent inductance 0.4 H when connected in opposition. Calculate the mutual inductance.*

LOTS

Solution :

$L_1 + L_2 + 2M = 0.8$ H ... series aiding

$L_1 + L_2 - 2M = 0.4$ H ... series opposing

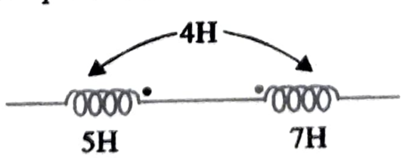
Solving we get

$4M = 0.4$ H

$M = 0.1$ H

$\therefore$ Mutual inductance $\boxed{M = 0.1 \text{ H}}$

EXAMPLE 45: *Determine the equivalent inductance of series combination.*



LOTS

Solution :

$L_1 = 5$ H, $L_2 = 7$ H, $M = 4$ H

The equivalent inductance is given by

$L_{eq} = L_1 + L_2 - 2M = 5 + 7 - 2 \times 4$

Equivalent inductance $\boxed{L_{eq} = 4 \text{ H}}$