Resonance and Coupled Circuits

4.1 RESONANCE IN SERIES AC CIRCUITS

Learning Objective (LO 1)

- ***Students will be able to analyze the resonance in series AC circuits with resonant frequency, bandwidth Q-factor and half power frequencies involved in it.***

Figure 4.1 shows the series RLC circuit. An *AC* circuit is said to be in resonance if it behaves in effect like a purely resistive circuit. The total current drawn by the circuit is in phase with the applied voltage and the power factor will then be unity.

The applied voltage $V = \overline{V_R} + \overline{V_L} + \overline{V_C}$

$$V = IR + jIX_L - jIX_C$$

$$V = I\,(R + jX_L - jX_C)$$

$$\frac{V}{I} = R + j\,(X_L - X_C)$$

$$\therefore \quad Z = R + j\,(X_L - X_C)$$

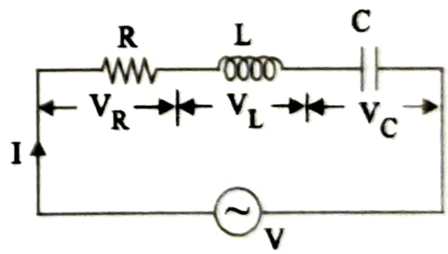


Fig. 4.1

Total impedance $Z = R + j\left(\omega L - \dfrac{1}{\omega C}\right)$

In a series RLC circuit, the current lags behind or leads the applied voltage depending upon the values of X_L and X_C. X_L causes the total current to lag behind the applied voltage, while X_C causes the total current to lead the applied voltage. When $X_L > X_C$ the circuit is predominantly inductive, and when $X_C > X_L$, the circuit is predominantly capacitive. The variations of X_L and X_C with respect to ω is shown in figure 4.2

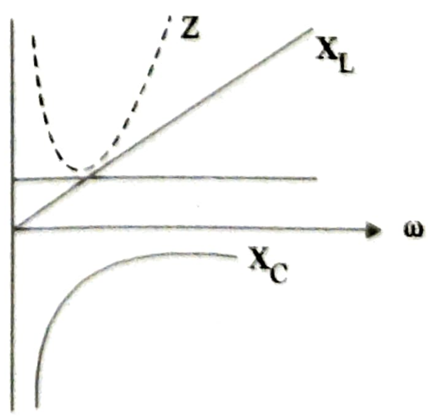


Fig. 4.2

At a certain frequency, $X_L = X_C$. When $X_L = X_C$, the circuit is said to be in resonance. At resonant condition, the total impedance Z is minimum and is equal to R. The circuit will be a purely resistive circuit.

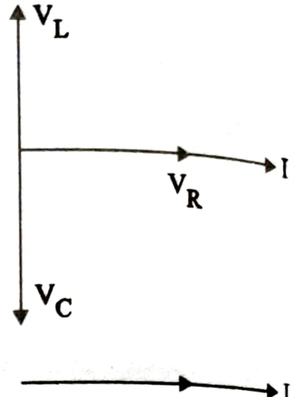


Fig. 4.3

Current $I = \frac{V}{R}$

$$V_L = IX_L \, ; \, V_C = I X_C$$

Since $X_L = X_C$, the two voltages are equal in magnitude and opposite in phase. Hence, they cancel out each other. The phasor diagram is shown in figure 4.3.

4.1.1 Resonant Frequency (f_r)

The frequency at which resonance occurs is called resonant frequency.

When $X_L = X_C$,

$$\omega_r L = \frac{1}{\omega_r C}$$

$$\omega_r^2 LC = 1$$

$$\omega_r^2 = \frac{1}{LC}$$

$$\omega_r = \frac{1}{\sqrt{LC}}$$

$$2\pi f_r = \frac{1}{\sqrt{LC}}$$

$$\boxed{f_r = \frac{1}{2\pi\sqrt{LC}}}$$

In a series RLC circuit, resonance may be produced by varying the frequency, keeping L and C constant; resonance may also be produced by varying either L or C for a fixed frequency as given in figure 4.4.

When $f < f_r$, X_c will be greater than X_L; $p.f$ is leading.

When $f > f_r$, X_L will be greater than X_C; $p.f$ is lagging.

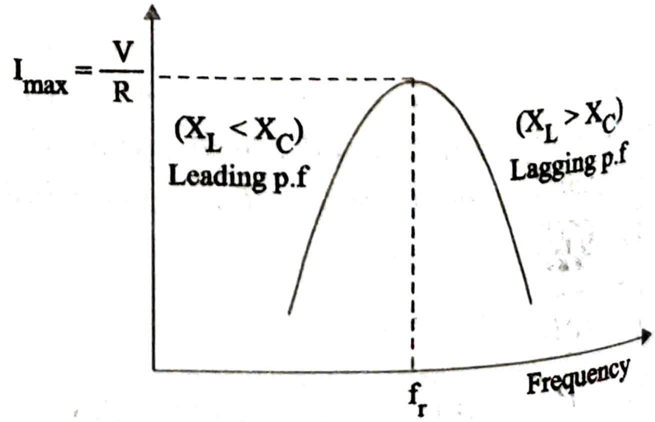


Fig. 4.4

EXAMPLE 1: *A resistor of 50 Ω, an inductor of 0.02 H and a capacitor of 5 μF are connected in series. Find the resonant frequency and power factor at resonance.*

LOTS

Solution :

$R = 5\ \Omega,\ L = 0.02\ \text{H},\ C = 5\ \mu\text{F} = 5 \times 10^{-6}\ \text{F}$

$$\text{Resonant frequency } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.02 \times 5 \times 10^{-6}}}$$

$$\boxed{f_r = 503.292\ \text{Hz}}$$

Under resonance condition, power factor $\boxed{\cos\phi = 1}$

EXAMPLE 2: *A coil of resistance 2 Ω and inductance 0.01 H is connected in series with a capacitor C. If maximum current occurs at 25 Hz, find C.*

LOTS

Solution :

When the current is maximum the circuit is at resonance. Therefore, the given frequency is resonant frequency.

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$f_r^2 = \frac{1}{(2\pi)^2 LC}$$

$$C = \frac{1}{(2\pi)^2 \times f_r^2 \times L} = \frac{1}{4\pi^2 \times (25)^2 \times 0.01}$$

$$= 4052 \times 10^{-6}\ \text{F}$$

$$\boxed{C = 4052\ \mu\text{F}}$$

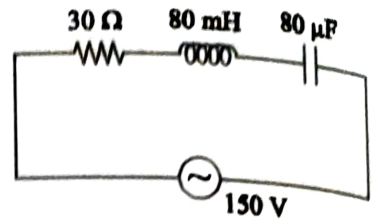
EXAMPLE 3: *A series RLC circuit has R = 30 Ω, L = 80 mH and C = 80 μF. Find the resonant frequency. Obtain the current, power, the voltage drops across the various elements if the applied voltage equals 150 volts.*

HOTS

Solution :

$R = 30\ \Omega\ L = 80\ \text{mH},\ C = 80\ \mu\text{F},\ V = 150\ \text{V}$

HOTS – Higher Order Thinking Skills LOTS – Lower Order Thinking Skills



Resonant frequency $f_r = \frac{1}{2\pi\sqrt{LC}}$

$$= \frac{1}{2\pi\sqrt{80 \times 10^{-3} \times 80 \times 10^{-6}}}$$

$$\boxed{f_r = 63 \text{ Hz}}$$

At resonance, Current $I = \frac{V}{R} = \frac{150}{30} = 5$ A

$$\boxed{I = 5 \text{ A}}$$

Power $P = I^2R = (5)^2 \times 30 = 750$ W

Voltage across the resistor $V_R = IR = 5 \times 30 = 150$ V

$$\boxed{V_R = 150 \text{ V}}$$

Voltage across the inductor $V_L = IX_L = I\,(2\pi fL)$

$$V_L = 5 \times 2\pi \times 63 \times 80 \times 10^{-3} = 158 \text{ volts}$$

$$\boxed{V_L = 158 \text{ V}}$$

Voltage across the capacitor $V_C = IX_C = I\left(\frac{1}{2\pi fC}\right)$

$$V_C = 5 \times \frac{1}{2\pi \times 63 \times 80 \times 10^{-6}}$$

$$\boxed{V_C = 158 \text{ V}}$$

EXAMPLE 4: ***A series RLC circuit is connected to a 220 V, 50 Hz supply, when L is varied, the maximum current obtained is 0.4 ampere and the voltage across the capacitor then is 330 volts. Find the circuit constants.***

LOTS

Solution :

Supply voltage = 220 V

Frequency = 50 Hz

Maximum current = 0.4 A

Voltage across capacitor $V_c = 330$ V

Maximum current corresponds to resonant condition

Capacitive reactance $X_C = \frac{V_C}{I} = \frac{330}{0.4} = 825\ \Omega$

$$X_C = \frac{1}{2\pi f C}$$

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi \times 50 \times 825}$$

$$C = 3.858 \times 10^{-6}\ \text{F}$$

$$\boxed{C = 3.858\ \mu\text{F}}$$

At resonance $X_L = X_C$

$$\therefore\ X_L = 825\ \Omega$$

$$X_L = 2\pi f L$$

$$L = \frac{X_L}{2\pi f} = \frac{825}{2\pi \times 50}$$

$$\boxed{L = 2.626\ \text{H}}$$

At resonance $Z = R$

$$R = \frac{V}{I_{max}} = \frac{220}{0.4}$$

$$\boxed{R = 550\ \Omega}$$

4.1.2 Voltage and current in a series resonant circuit

Figure 4.5 shows the variation of impedance and current with frequency. At resonance, the reactance, and hence the impedance is minimum. Because of minimum impedance, maximum current flows through the circuit. The voltage drops across resistor, inductor and capacitor also vary with frequency. At $f = 0$, the capacitor acts as an open circuit and blocks current. The complete source voltage appears across the capacitor. As the frequency increases, X_C decreases and X_L increases, causing total reactance $X_C - X_L$ to decrease. As a result, the total impedance of the circuit decreases and the current increases. As the current increases, V_R, V_L and V_C also increase. When the frequency reaches its resonant value f_r, the impedance is equal to R, and hence, the current reaches its maximum value and V_R is at its maximum value.

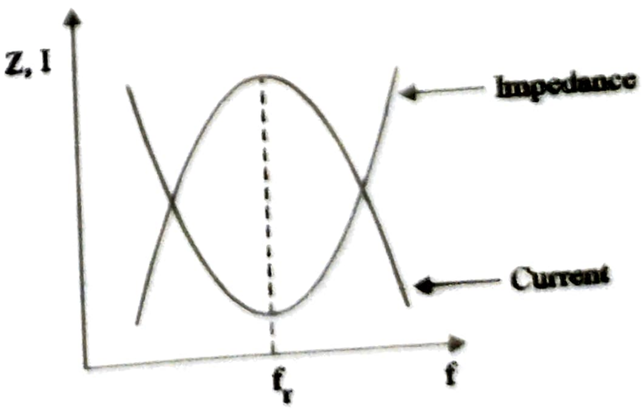


Fig. 4.5

4.1.4 Half power frequencies

The upper and lower cut-off frequencies are sometimes called the half power frequencies. At these frequencies, the power from the source is half of the power delivered at the resonant frequency.

At resonant frequency, the power is $P_{max} = I_{max}^2 R$

At frequency f_1, the power is $P_1 = \left(\frac{I_{max}}{\sqrt{2}}\right)^2 R$

$$P_1 = \frac{I_{max}^2 R}{2}$$

At frequency f_2, the power is $P_2 = \left(\frac{I_{max}}{\sqrt{2}}\right)^2 R$

$$P_2 = \frac{I_{max}^2 R}{2}$$

4.1.5 Selectivity

The ratio of bandwidth to the resonant frequency is defined as the selectivity.

$$\text{Selectivity} = \frac{\text{Bandwidth}}{\text{Resonant frequency}}$$

$$\text{Selectivity} = \frac{f_2 - f_1}{f_r}$$

$$\text{Selectivity} = \frac{R}{2\pi f_r L} = \frac{R}{\omega_r L} = \frac{1}{Q}$$

where Q is the quality of the circuit.

Note:

$$Q = \frac{\omega_r L}{R}$$

$$= \frac{1}{\sqrt{LC}} \cdot \frac{L}{R}$$

$$= \frac{1}{R}\sqrt{L/C}$$

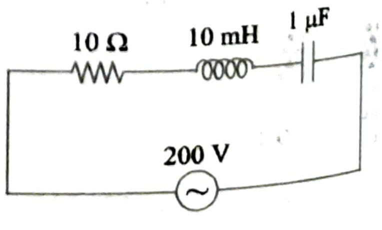
EXAMPLE 5: *A series RLC circuit with R = 10 ohms, L = 10 mH and C = 1 μF has an applied voltage of 200 V at resonant frequency. Calculate the resonant frequency, the current in the circuit and voltages across the elements at resonance. Find also the quality factor and bandwidth.*

HOTS (AU, Coimbatore/EEE - May 2011), (AU, Chennai/EEE - June 2010)

Solution :

$R = 10\Omega$, $L = 10$ mH , $C = 1$ μF

Supply voltage = 200 V



Resonant frequency $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{10\times10^{-3}\times1\times10^{-6}}}$

$$\boxed{f_r = 1591.5494 \text{ Hz}}$$

At resonance, current $I_{max} = \frac{V}{R} = \frac{200}{10} = 20$ A

Voltage across resistor $V_R = I_{max} R = 20\times10 = 200$ Volts

$$\boxed{V_R = 200 \text{ V}}$$

Voltage across inductor $V_L = I_{max} X_L = I_{max}(2\pi f L)$

$$V_L = 20\times2\pi\times1591.5494\times10\times10^{-3}$$

$$\boxed{V_L = 2000 \text{ Volts}}$$

Voltage across capacitor $V_C = I_{max} X_C = I_{max}\left(\frac{1}{2\pi f C}\right)$

$$V_C = 20\times\frac{1}{2\pi\times1591.5494\times1\times10^{-6}}$$

$$\boxed{V_C = 2000 \text{ volts}}$$

Quality factor $Q = \frac{\omega_r L}{R}$

$$Q = \frac{2\pi\times1591.5494\times10\times10^{-3}}{10}$$

$$\boxed{Q = 10}$$

Bandwidth $= \frac{\text{Resonant frequency}}{\text{Quality factor}}$

$$BW = \frac{f_r}{Q} = \frac{1591.5494}{10}$$

$$\boxed{BW = 159.15 \text{ Hz}}$$

EXAMPLE 6: *What is the resonant frequency and bandwidth of a series RLC circuit given that $R = 5\Omega$, $L = 40$ mH, $C = 1$ μF.*

LOTS

Solution :

$R = 5\Omega$, $L = 40$ mH , $C = 1$ μF

$$\text{Resonant frequency } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{40\times 10^{-3}\times 1\times 10^{-6}}}$$

$$\boxed{f_r = 795.774 \text{ Hz}}$$

$$\text{Quality factor } Q = \frac{\omega_r L}{R}$$

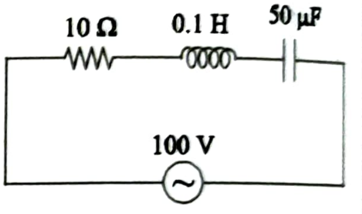
$$Q = \frac{2\pi f_r L}{R} = \frac{2\pi\times 795.774\times 40\times 10^{-3}}{5}$$

$$\boxed{Q = 40}$$

$$\text{Bandwidth } BW = \frac{f_r}{Q} = \frac{795.774}{40}$$

$$\boxed{BW = 19.8943 \text{ Hz}}$$

EXAMPLE 7: *For the circuit shown in figure, determine the frequency at which the circuit resonates. Also find the voltage across the inductor at resonance and the Q factor of the circuit.*



LOTS

Solution :

$$\text{Resonant frequency } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.1\times 50\times 10^{-6}}}$$

$$\boxed{f_r = 71.176 \text{ Hz}}$$

$$\text{Current at resonance } I = \frac{V}{R} = \frac{100}{10} = 10 \text{ A}$$

$$\boxed{I = 10 \text{ A}}$$

The voltage drop across the inductor $V_L = IX_L$

$$V_L = I\times 2\pi f_r L = 10\times 2\pi\times 71.176\times 0.1$$

$$\boxed{V_L = 447.21 \text{ Volts}}$$

$$\text{Quality factor } Q = \frac{\omega_r L}{R}$$

$$Q = \frac{2\pi f_r L}{R} = \frac{2\pi\times 71.176\times 0.1}{10} = 4.4721$$

$$\boxed{Q = 4.4721}$$

EXAMPLE 8: *Determine the Quality factor of a coil for the series circuit consisting of $R = 10\Omega$, $L = 0.1$ H, $C = 10$ μF.*

LOTS

(AU, Chennai/EEE - Dec. 2010)

Solution :

$R = 10\Omega$, $L = 0.1$ H , $C = 10$ μF

$$\text{Quality factor } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.1}{10 \times 10^{-6}}}$$

$$\boxed{Q = 10}$$

EXAMPLE 9: *Compute Q of the series RLC circuit with $R = 10\ \Omega$, $L = 0.04$ H and $C = 1$ μF. Find Bandwidth, resonant frequency and half power frequencies.*

HOTS

(AU, Madurai/EEE - May 2011)

Solution :

$R = 10\ \Omega$, $L = 0.04$ H, $C = 1$ μ F

$$\text{Quality factor } Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.04}{1 \times 10^{-6}}}$$

$$\boxed{Q = 20}$$

$$\text{Bandwidth } = f_2 - f_1 = \frac{R}{2\pi L} = \frac{10}{2\pi \times 0.04} = 39.78 \text{ Hz}$$

$$\boxed{BW = 39.78 \text{ Hz}}$$

$$\text{Resonant frequency } f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.04 \times 10^{-6}}}$$

$$\boxed{f_r = 795.77 \text{ Hz}}$$

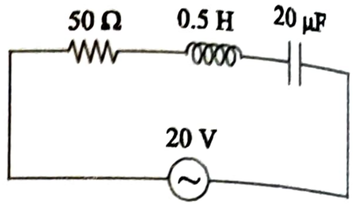
$$\text{Half power frequency } f_1 = f_r - \frac{R}{4\pi L} = 795.77 - \frac{39.78}{2}$$

$$\boxed{f_1 = 775.88 \text{ Hz}}$$

Half power frequency $f_2 = f_r + \frac{R}{4\pi L} = 795.77 + \frac{39.78}{2}$

$$\boxed{f_2 = 815.6 \text{ Hz}}$$

EXAMPLE 10: *For the circuit shown in figure, determine the frequency at which the circuit resonates. Also, find the voltage across the capacitor at resonance and the Q factor of the circuit.*



HOTS

Solution :

$V = 20$ V , $R = 50\ \Omega$, $L = 0.5$ H , $C = 20\ \mu$F

Circuit current $I = \frac{V}{R} = \frac{20}{50} = 0.4$ A

Resonant frequency $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.5 \times 20 \times 10^{-6}}}$

$$\boxed{f_r = 50.33 \text{ Hz}}$$

Voltage across the capacitor $V_c = IX_c$

$$V_c = \frac{I}{\omega_r C}$$

$$V_c = \frac{I}{2\pi f_r C} = \frac{0.4}{2\pi \times 50.33 \times 20 \times 10^{-6}}$$

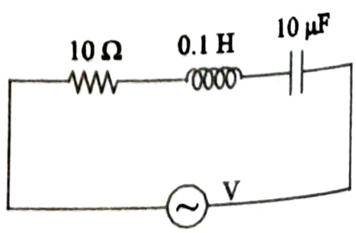
$$\boxed{V_c = 63.244 \text{ Volts}}$$

Quality factor $Q = \frac{\omega_r L}{R}$

$$Q = \frac{2\pi f_r L}{R} = \frac{2\pi \times 50.33 \times 0.5}{50}$$

$$\boxed{Q = 3.1622}$$

EXAMPLE 11: *For the circuit shown in figure, determine the impedance at resonant frequency, 10 Hz above resonant frequency, and 10 Hz below resonant frequency.*



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Solution :

Resonant frequency $f_r = \frac{1}{2\pi\sqrt{LC}}$

$$f_r = \frac{1}{2\pi\sqrt{0.1 \times 10 \times 10^{-6}}} = 159.1549 \text{ Hz}$$

At 10 Hz below $f_r = 159.2 - 10 = 149.2$ Hz

At 10 Hz above $f_r = 159.2 + 10 = 169.2$ Hz

Impedance at resonance is equal to R

$$Z = 10\ \Omega$$

Capacitive reactance at 149.2 Hz is

$$X_{c1} = \frac{1}{\omega_1 C} = \frac{1}{2\pi f_1 C}$$

$$X_{c1} = \frac{1}{2\pi \times 149.2 \times 10 \times 10^{-6}} = 106.672\ \Omega$$

Capacitive reactance at 169.2 Hz is

$$X_{C2} = \frac{1}{\omega_2 C} = \frac{1}{2\pi f_2 C}$$

$$X_{C2} = \frac{1}{2\pi \times 169.2 \times 10 \times 10^{-6}} = 94.06\ \Omega$$

Inductive reactance at 149.2 Hz is

$$X_{L1} = \omega_1 L = 2\pi f_1 L = 2\pi \times 149.2 \times 0.1 = 93.745\ \Omega$$

Inductive reactance at 169.2 Hz is

$$X_{L2} = \omega_2 L = 2\pi f_2 L = 2\pi \times 169.2 \times 0.1 = 106.31\ \Omega$$

Impedance at 149.2 Hz is

$$Z_1 = \sqrt{R^2 + (X_{L1} - X_{C1})^2} = \sqrt{(10)^2 + (93.745 - 106.6)^2}$$

$$\boxed{Z_1 = 16.286\ \Omega}$$

Impedance at 169.2 Hz is

$$Z_2 = \sqrt{R^2 + (X_{L2} - X_{C2})^2} = \sqrt{(10)^2 + (106.31 - 94.06)^2}$$

$$\boxed{Z_2 = 15.813\ \Omega}$$