SNS COLLEGE OF ENGINEERING
Kurumbapalayam (Po), Coimbatore – 641 107
AN AUTONOMOUS INSTITUTION
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Topic: 3. 9 – ENVELOPES

Problems – Envelope of two parameter family of curves.

1. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters a and b are connected by the relation $a + b = c$.

soln :

Given $\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow$ ①

Taking differential we get.

$$-\frac{x}{a^2} da - \frac{y}{b^2} db = 0 \longrightarrow ②$$

Given $a + b = c \rightarrow$ ③

Taking differential we get

$$da + db = 0 \rightarrow ④$$

② $\Rightarrow \quad \frac{x}{a^2} da = -\frac{y}{b^2} db$

④ $\Rightarrow \quad da = -db$

$\frac{②}{④} \Rightarrow \quad \dfrac{\frac{x}{a^2} da}{da} = \dfrac{-\frac{y^2}{b^2} db}{-db}$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\frac{\frac{x}{a}}{a} = \frac{\frac{y}{b}}{b} = \frac{\frac{x}{a} + \frac{y}{b}}{a + b} = \frac{1}{c}$$

(i.e) $\frac{x}{a^2} = \frac{1}{c}$; $\frac{y}{b^2} = \frac{1}{c}$

$\Rightarrow a^2 = xc$; $b^2 = yc$.

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$$a = (xc)^{1/2} \quad ; \quad b = (yc)^{1/2}$$

$\textcircled{3} \Rightarrow$
$$a + b = c$$
$$(xc)^{1/2} + (yc)^{1/2} = c$$
$$\sqrt{x}\sqrt{c} + \sqrt{y}\sqrt{c} = c$$
$$\sqrt{c}\left[\sqrt{x} + \sqrt{y}\right] = c$$
$$\sqrt{x} + \sqrt{y} = \frac{c}{\sqrt{c}}$$
$$\sqrt{x} + \sqrt{y} = \sqrt{c}$$

2. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$ where the parameters a and b are connected by the relation $a^2 + b^2 = c^2$

soln:

Given $\frac{x}{a} + \frac{y}{b} = 1 \longrightarrow \textcircled{1}$

Taking differentials; $-\frac{x}{a^2}da - \frac{y}{b^2}db = 0 \rightarrow \textcircled{2}$

Given $a^2 + b^2 = c^2 \rightarrow \textcircled{3}$

Taking differential $2a\,da + 2b\,db = 0 \rightarrow \textcircled{4}$

$\textcircled{2} \Rightarrow \frac{x}{a^2}da = -\frac{y}{b^2}db \rightarrow \textcircled{5}$

$\textcircled{4} \Rightarrow 2a\,da = -2b\,db \rightarrow \textcircled{6}$

$\frac{\textcircled{5}}{\textcircled{6}} \Rightarrow \frac{x/a^2}{a} = \frac{y/b^2}{b} \Rightarrow \frac{x/a + y/b}{a^2 + b^2} = \frac{1}{c^2}$
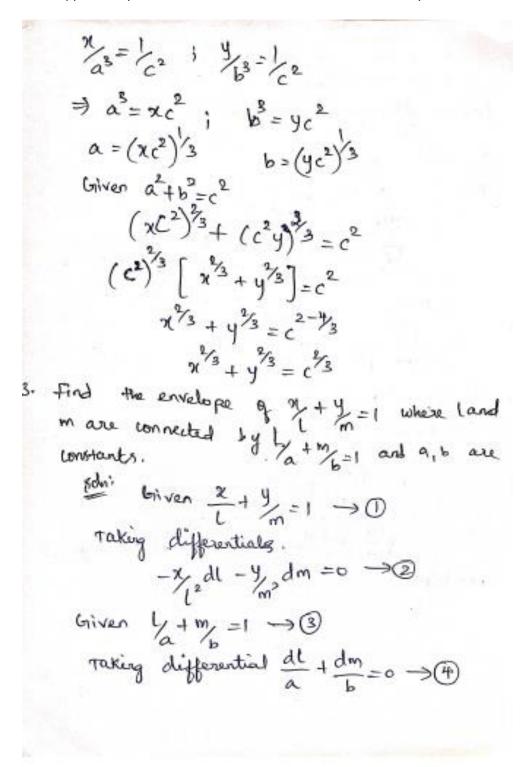
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$$\frac{x}{a^3} = \frac{1}{c^2} \quad ; \quad \frac{y}{b^3} = \frac{1}{c^2}$$

$$\Rightarrow a^3 = xc^2 \quad ; \quad b^3 = yc^2$$

$$a = (xc^2)^{1/3} \qquad b = (yc^2)^{1/3}$$

Given $a^2 + b^2 = c^2$

$$(xc^2)^{2/3} + (c^2 y)^{2/3} = c^2$$

$$(c^2)^{2/3} \left[x^{2/3} + y^{2/3} \right] = c^2$$

$$x^{2/3} + y^{2/3} = c^{2 - 4/3}$$

$$x^{2/3} + y^{2/3} = c^{2/3}$$

3. Find the envelope of $\frac{x}{l} + \frac{y}{m} = 1$ where l and m are connected by $\frac{l}{a} + \frac{m}{b} = 1$ and a, b are constants.

   Soln:

   Given $\frac{x}{l} + \frac{y}{m} = 1 \longrightarrow \text{①}$

   Taking differentials.

   $$-\frac{x}{l^2} dl - \frac{y}{m^2} dm = 0 \longrightarrow \text{②}$$

   Given $\frac{l}{a} + \frac{m}{b} = 1 \longrightarrow \text{③}$

   Taking differential $\frac{dl}{a} + \frac{dm}{b} = 0 \longrightarrow \text{④}$

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$$② ⇒ \frac{x}{l^2} dl = -\frac{y}{m^2} dm \rightarrow ⑤$$

$$④ ⇒ \frac{dl}{a} = -\frac{dm}{b} \rightarrow ⑥$$

$$\frac{⑤}{⑥} ⇒ \frac{ax}{l^2} = \frac{-by}{m^2}$$

$$\frac{x/l}{y/a} = \frac{-y/m}{m/b} = \frac{x/l + y/m}{y/a + m/b} = \frac{1}{1} = 1$$

$$\frac{xa}{l^2} = 1 \quad ; \quad \frac{yb}{m^2} = 1$$

$$xa = l^2 \qquad yb = m^2$$

$$(i-e) \quad l = \sqrt{ax} \quad ; \qquad y = \sqrt{by}$$

Given $\frac{l}{a} + \frac{m}{b} = 1$

$$\frac{\sqrt{ax}}{la} + \frac{\sqrt{ay}}{b} = 1$$

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1 \quad \text{which is the required}$$

envelope.