



Topic: 2.8 – Alternating series – Leibnitz's test

Alternating Series - Leibnitz's test:

Alternating series.
A series in which terms are alternately positive & negative is called an alternating series.
Eg: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

Leibnitz's rule:
An alternating series $u_1 - u_2 + u_3 - u_4 + \dots$
converges if (i) $u_n - u_{n-1} < 0$ (consider numerical value of u_n & u_{n-1})
(ii) $\lim_{n \rightarrow \infty} u_n = 0$ (consider numerical values of u_n)

1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$

Sol: The terms of the given series are alternately +ve & -ve.

Step:1 To find $u_n = \frac{1}{n}$
 $u_{n-1} = \frac{1}{n-1}$
 $u_n - u_{n-1} = \frac{1}{n} - \frac{1}{n-1}$
 $= \frac{n-1-n}{n(n-1)} = \frac{-1}{n(n-1)} < 0$
 $\Rightarrow u_n - u_{n-1} < 0$ — (1)

Step:2 To find $\lim_{n \rightarrow \infty} u_n$
 $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ — (2)

Step:3 conclusion:
From (1) & (2) Leibnitz's rule satisfied.
 \therefore The given series is convergent.



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

2. Discuss the convergence of the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \dots$

Solve:
The terms of the given series are alternately +ve & -ve.

Step:1 To find u_n & u_{n-1} (numerically)
Here $u_n = \frac{1}{\sqrt{n}}$
 $u_{n-1} = \frac{1}{\sqrt{n-1}}$
 $u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} = \frac{\sqrt{n-1} - \sqrt{n}}{\sqrt{n}\sqrt{n-1}} < 0$
 $\Rightarrow u_n - u_{n-1} < 0$ ——— ①

Step:2 To find $\lim_{n \rightarrow \infty} u_n$
 $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$ ——— ②

Step:3: From ① & ② Leibnitz's rule satisfied
 \therefore the given series is convergent.

3. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2n-1}$

Solve:
The terms of the given series are alternately +ve and -ve.
Here $u_n = \frac{n}{2n-1}$, $u_{n-1} = \frac{n-1}{2(n-1)-1} = \frac{n-1}{2n-3}$
 $u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3}$
 $= \frac{2n^2 - 3n - 2n^2 + 2n + 2n - 1}{(2n-1)(2n-3)} = \frac{-1}{(2n-1)(2n-3)} < 0$ ——— ①
 $\Rightarrow u_n - u_{n-1} < 0$.
 $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \lim_{n \rightarrow \infty} \frac{1}{2 - \frac{1}{n}} = \frac{1}{2} \neq 0$ ——— ②

From ① & ② Leibnitz's rules are not satisfied,
 \therefore the given series is not convergent.
 \Rightarrow Hence the given series is oscillatory.



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

4. Examine the character of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum \frac{(-1)^n}{n^2+1}$

Soln:
The terms of the given series are alternately +ve and negative.

Here $u_n = \frac{1}{n^2+1}$
 $u_{n-1} = \frac{1}{(n-1)^2+1} = \frac{1}{n^2-2n+1+1}$
 $= \frac{1}{n^2-2n+2}$
 $u_n - u_{n-1} = \frac{1}{n^2+1} - \frac{1}{n^2-2n+2}$
 $= \frac{n^2-2n+2-n^2-1}{(n^2+1)(n^2-2n+2)}$
 $= \frac{-2n+1}{(n^2+1)(n^2-2n+2)} < 0$
 $\Rightarrow u_n - u_{n-1} < 0 \quad \text{--- (1)}$

$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0 \quad \text{--- (2)}$

from (1) & (2) Leibnitz's rules are satisfied
 \therefore the given series is convergent.

5. Discuss the convergence of the series
 $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \dots \quad 0 < x < 1.$

Soln:
The terms of the given series are alternately positive and negative.

Here $u_n = \frac{x^n}{1+x^n}$, $u_{n-1} = \frac{x^{n-1}}{1+x^{n-1}}$
 $u_n - u_{n-1} = x^n \left[\frac{1}{1+x^n} - \frac{1}{1+x^{n-1}} \right]$



$$= x^n \left[\frac{x+x^n - 1-x^n}{(1-x^n)(x+x^n)} \right]$$

$$= x^n \left[\frac{x-1}{(1+x^n)(x+x^n)} \right] < 0 \quad [\because 0 < x < 1]$$

$$\Rightarrow u_n - u_{n-1} < 0 \quad \text{--- (1)}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{x^n}{1+x^n} = 0 \quad \text{--- (2)}$$

From (1) & (2) Leibnitz's rule satisfied.
 \therefore The given series is convergent.

6. Discuss the convergence of the series
 $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) - \dots$

Solve - The terms of the given series are alternately positive and negative.
 Here $u_n = \frac{1}{2} - \frac{1}{\log(n)}$, $u_{n-1} = \frac{1}{2} - \frac{1}{\log(n-1)}$

$$u_n - u_{n-1} = \left[\frac{1}{2} - \frac{1}{\log(n)} \right] - \left[\frac{1}{2} - \frac{1}{\log(n-1)} \right]$$

$$= \frac{1}{\log(n-1)} - \frac{1}{\log(n)} = \frac{\log(n-1) - \log(n)}{\log(n-1)\log(n)} < 0$$

$$\Rightarrow u_n - u_{n-1} < 0 \quad \text{--- (1)}$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2} - \frac{1}{\log(n)} \right] = \frac{1}{2} \neq 0 \quad \text{--- (2)}$$

From (1) & (2) both the conditions of Leibnitz's rule not satisfied.
 Hence the series is oscillatory and it oscillates b/w $-\infty$ and ∞ .